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Pseudo-free action of cyclic groups

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1 Determinant of equivariant elliptic operators

$M = M^{2m}$: $2m$-dimensional closed connected oriented Riemannian manifold

G : finite group acting on M

G-action is orientation-preserving, isometric and effective

$D : \Gamma(E) \to \Gamma(F)$: G-equivariant elliptic operator

Equivariant determinant of D is defined by

$$G \ni g \longrightarrow \det(D, g) = \frac{\det(g|\ker D)}{\det(g|\operatorname{coker} D)} \in S^1 \subset \mathbf{C}^*$$

$\det_D := \det(D, \cdot\,) : G \longrightarrow S^1$ is a group homomorphism

S^1 : Abelian $\Longrightarrow$ $\det_D([G, G]) = 1$

I_D : $G \longrightarrow \mathbf{R}/\mathbf{Z}$ (additive group homomorphism) is defined by

$$I_D(g) := \frac{1}{2\pi\sqrt{-1}} \log\det(D, g) \pmod{\mathbf{Z}}$$

<u>**Theorem 1**</u> I_D is an additive group homomorphism and hence

(a) $I_D(g) + I_D(h) - I_D(gh) = 0$ $(\forall g,\, h \in G)$

(b) $\det(D, g)^N = 1 \iff N\, I_D(g) = 0$

<u>**Proposition 1**</u> $g^p = 1$ $(p \geq 2)$ $\Longrightarrow$

$$I_D(g) \equiv \frac{p-1}{2p}\mathrm{Ind}(D) - \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\mathrm{Ind}(D, g^k) \pmod{\mathbf{Z}}$$

where $\xi_p = e^{2\pi\sqrt{-1}/p}$: primitive p-th root of unity,

$\mathrm{Ind}(D, g^k) = \mathrm{Tr}(g^k|\ker D) - \mathrm{Tr}(g^k|\mathrm{coker}\, D) \in \mathbf{C}$,

$\mathrm{Ind}(D) = \mathrm{Ind}(D, 1) = \dim\ker D - \dim\mathrm{coker}\, D \in \mathbf{Z}$

$\mathrm{Ind}(D)$, $\mathrm{Ind}(D, h)$ $(h \in G)$ and hence $I_D(h)$ is calculated by using the Atiyah-Singer index theorem.

$\mathrm{Ind}(D)$ is a characteristic number defined by using

the principal symbol of D

$$\begin{cases} M \supset \mathrm{Fix}(g) = \{q_1, q_2, \cdots, q_n\} \text{ consists of points} \\ T_{q_i}M = \oplus_{j=1}^m N(\tau_{ij}) \quad \left(h|N(\tau) = \begin{pmatrix} \cos\frac{2\pi\tau}{p} & -\sin\frac{2\pi\tau}{p} \\ \sin\frac{2\pi\tau}{p} & \cos\frac{2\pi\tau}{p} \end{pmatrix} \right) \end{cases}$$

$$\Longrightarrow \mathrm{Ind}(D, h) = \sum_{i=1}^n \frac{\mathrm{Tr}(h|E_{q_i}) - \mathrm{Tr}(h|F_{q_i})}{\prod_{j=1}^m (1-\xi_p^{\tau_{ij}})(1-\xi_p^{-\tau_{ij}})}$$

<u>**Proposition 2**</u> Let S be the signature operator and assume that p is an odd prime number. Then we have

$$\mathrm{Ind}(S) = \mathrm{Sign}(M), \quad \mathrm{Ind}(S, h) = \sum_{i=1}^n \prod_{j=1}^m \left(-\sqrt{-1}\cot\frac{\pi\tau_{ij}}{p}\right)$$

where $\mathrm{Sign}(M)$ is the signature of M.

$\mathrm{Spin}(2m)$: universal covering group of $SO(2m)$ $(m \geq 2)$

$\mathrm{Spin}^c(2m) := \mathrm{Spin}(2m) \times S^1/\sim \quad ((s, z) \sim (-s, -z))$

$\exists P$: Spin^c-structure of $M \Longleftrightarrow w_2(TM) = \exists u \in H^2(M; \mathbf{Z}) \pmod 2$

$\exists$ Spin or $\exists$ almost complex structure $\Longrightarrow$ $\exists\mathrm{Spin}^c$-structure

$\eta = P \times_{\mathrm{Spin}^c(2m)} \mathbf{C}$: associated complex line bundle over M

(Spin^c-structure is determined by $c_1(\eta)$ if $H^1(M; \mathbf{Z}_2) = 0$)

G-action on a Spin^c-manifold M is called a <u>Spin^c-action</u>

if the action lifts to an action on the Spin^c-structure of M

<u>**Remark 1**</u> Let G be a cyclic group and assume that $m \geq 2$. Then
if $H^1(M;\mathbf{Z}) = 0 (\Longrightarrow H^1(\mathrm{SO}(M);\mathbf{Z}) = 0)$ and $c_1(\eta)$ is G-invariant

$\Longrightarrow$ any G-action lifts to the Spin^c-structure $P(\xrightarrow{S^1} \mathrm{SO}(M))$.

(A.Hattori and T.Yoshida, Lifting compact group actions in fiber bundles, Japan. J. Math. **2**, 13-25 (1976).)

L : (arbitrary) complex G-line bundle over M

$$D_L : \Gamma(S_+ \otimes L) \xrightarrow{\nabla} \Gamma(T^*M \otimes S_+ \otimes L) \xrightarrow{cm} \Gamma(S_- \otimes L) : L\text{-valued Dirac operator}$$

(where $S_\pm$ are half spinor bundles and cm is the Clifford multiplication)

<u>**Proposition 3**</u>

h acts on the fibers $\eta|q_i$, $L|q_i$ via multiplications by $\xi_p^{\kappa_i}$, $\xi_p^{\mu_i}$ respectively

$$\Longrightarrow \mathrm{Ind}(D_L) = e^{c_1(L)} e^{\frac{c_1(\eta)}{2}} \widehat{A}(TM)[M] \ ,$$
$$\mathrm{Ind}(D_L, h) = \sum_{i=1}^{n} \varepsilon_i \xi_p^{\mu_i} \xi_p^{\frac{\nu_i}{2}} \prod_{j=1}^{m} \frac{1}{1 - \xi_p^{-\tau_{ij}}}$$

where $\widehat{A}$ is the $\widehat{A}$-class, $\varepsilon_i = \pm 1$ and $\nu_i = \kappa_i - \sum_{j=1}^m \tau_{ij}$.
(ε , κ_i are not determined by the data of the action on M)

<u>**Proposition 4**</u> $M = (M, J)$: almost complex manifold
$G \subset \mathrm{Aut}_J(M)$: group of automorphisms of M preserving
the almost complex structure J
$h|T_{q_i}M = (\xi_p^{\tau_{i1}}, \cdots, \xi_p^{\tau_{im}})$: diagonal matrix with respect to J
h acts on the fiber $L|q_i$ via multiplication by $\xi_p^{\mu_i}$

$$\Longrightarrow \mathrm{Ind}(D_L) = e^{c_1(L)} \mathrm{Td}(TM)[M] \ , \quad \mathrm{Ind}(D_L, h) = \sum_{i=1}^{n} \xi_p^{\mu_i} \prod_{j=1}^{m} \frac{1}{1 - \xi_p^{-\tau_{ij}}}$$

where Td is the Todd class.

2 Finite subgroup of the mapping class group

Γ_σ : mapping class group of genus σ ($\sigma \geq 2$)

Σ_σ : compact Riemann surface of genus σ ($\sigma \geq 2$)

In this section, G-action $\Longleftrightarrow \exists J\,;\, G \subset \mathrm{Aut}_J(\Sigma_\sigma) \overset{(*)}{\Longleftrightarrow} G \subset \Gamma_\sigma$.

(J : (integrable) almost complex structure, $(*)$: Nielsen realization)

$\mathbf{Z}_p = \langle g \rangle \subset \mathrm{Aut}_J(\Sigma_\sigma)$

$\pi : \Sigma_\sigma \longrightarrow \Sigma_\sigma/\mathbf{Z}_p$: branched covering with b branch point

$y_1, \cdots, y_b \in \Sigma_\sigma/\mathbf{Z}_p$ of order $(n_1, \cdots, n_b)$

$\pi^{-1}(y_i) = \{q_i\,,\, g\cdot q_i\,,\, \cdots\,,\, g^{r_i-1}\cdot q_i\}$: consists of $r_i := p/n_i$ points

For $1 \leq i \leq b$, assume that $g^{r_i}|T_{\pi^{-1}(y_i)}\Sigma_\sigma = \xi_{n_i}^{t_i}$ where $1 \leq t_i \leq n_i - 1$ and t_i is prime to n_i

D_ℓ : $\otimes^\ell T\Sigma_\sigma$-valued Dirac operator on Σ_σ

Theorem 2

$$I_{D_\ell}(g^z) - z\, I_{D_\ell}(g) = 0 \iff \varphi_{\ell,z}(t_1, \cdots, t_b) \in \mathbf{Z}\,,$$
$$N\, I_{D_\ell}(g^z) = 0 \iff N\psi_{\ell,z}(t_1, \cdots, t_b) \in \mathbf{Z}$$

for any z ($1 \leq z < p$) which is prime to p and for any ℓ ($0 \leq \ell < p$) where

$$\begin{aligned}
&\varphi_{\ell,z}(t_1, \cdots, t_b) \\
&\quad = (1-z)\frac{p-1}{2p}(1-\sigma)(2\ell+1) \\
&\qquad - \sum_{i=1}^{b} \frac{1}{n_i} \sum_{j=1}^{n_i-1} \frac{1}{1-\xi_{n_i}^{-j}} \left(\frac{\xi_{n_i}^{jzt_i\ell}}{1-\xi_{n_i}^{-jzt_i}} - z\frac{\xi_{n_i}^{jt_i\ell}}{1-\xi_{n_i}^{-jt_i}} \right), \\
&I_{D_\ell}(g^k) \overset{\mathrm{mod.Z}}{\equiv} \psi_{\ell,z}(t_1, \cdots, t_b) \\
&\quad = \frac{p-1}{2p}(1-\sigma)(2\ell+1) - \sum_{i=1}^{b} \frac{1}{n_i} \sum_{j=1}^{n_i-1} \frac{\xi_{n_i}^{jzt_i\ell}}{(1-\xi_{n_i}^{-j})(1-\xi_{n_i}^{-jzt_i})}
\end{aligned}$$

and N is a natural number such that $\det(D_\ell, g)^N = 1$.

Precise values of φ and ψ are obtained by using the following proposition.

Proposition 5

$12p\, I_{D_\ell}(g^z)$ is an integer and we have

$$
\begin{aligned}
&12p\, I_{D_\ell}(g^z)\\
&\equiv 6(p-1)(1-\sigma)(2\ell+1)\\
&\quad + \sum_{i=1}^{b} r_i \left\{ zt_i(n_i-1)(7n_i-11) + 6 \sum_{j=\left[\frac{(\ell+1)zt_i}{n_i}\right]+1}^{\left[\frac{(\ell+n_i+1)zt_i}{n_i}\right]} f_{n_i}\left(\left[\frac{jn_i-1}{zt_i} \right] - \ell - 1 \right) \right\}\\
&\quad \pmod{12p}
\end{aligned}
$$

where $f_{n_i}(x) = x^2 - (n_i-2)x - (n_i-1)^2$ and [ ] is the Gauss's symbol.

Riemann-Hurwitz equation

$$2\sigma - 2 = p(2\overline{\sigma} - 2) + \sum_{i=1}^{b}(p - r_i) \quad \left(r_i = \frac{p}{n_i} \right)$$

where $\overline{\sigma}$ is the genus of $\Sigma_\sigma / \mathbf{Z}_p$.

Example 1 The necessary and sufficient condition of (σ, p) for $\mathbf{Z}_p \subset \Gamma_\sigma$ is known. (W.J.Harvey, Cyclic groups of automorphisms of a compact Riemann surface, Quart. J. Math. **17**, 86-97 (1966))
We consider one hundred cases where $2 \le \sigma,\ p \le 11$.
Then $\mathbf{Z}_p \not\subset \Gamma_\sigma$ if and only if

$(\sigma, p) = (2,7),\ (2,11),\ (3,11),\ (4,11),\ (5,7),\ (7,11),\ (8,11),\ (9,11)$

$\Longrightarrow$ R-H equation is not satisfied for any $\overline{\sigma}$, b, r_i

or

$(*1)\quad (\sigma, p) = (2,9),\ (3,5),\ (3,10),\ (4,7),\ (5,9),\ (6,11),\ (11,7)$

(Results of Harvey)

Using the R-H equation and Theorem 2, we can show that $\mathbf{Z}_p \not\subset \Gamma_\sigma$ for (σ, p) in $(*1)$.

Example 2

G_p : finite non-Abelian group generated by $g,\ h_1,\ \cdots,\ h_s$

order$(g) = p$, order$(h_j) = q_j$ $(1 \le j \le s)$

Assume that $[G_p, G_p] \ni \exists\gamma$ such that γ is expressed as the product

of m g's $(0 < m < p)$ and μ_j h_j's $(0 \le \mu_j < q_j,\ 1 \le j \le s)$

which satisfies that $d :=$ g.c.d.$(p,\ m\beta) < p$

(where $\beta := \ell.c.m(q_1^{\varepsilon_1} \cdots q_s^{\varepsilon_s})$, $\varepsilon_j = 1$ if $\mu_j > 0$, $\varepsilon_j = 0$ if $\mu_j = 0$)

Assume moreover that $\mathrm{Aut}_J(\Sigma_\sigma) \supset G_p$ $(2 \le \sigma \le 11\,,\ 3 \le p \le 11)$

$\Longrightarrow$ $\mathrm{Aut}_J(\Sigma_\sigma) \supset \mathbf{Z}_p = \langle g \rangle$

$\Sigma_\sigma \longrightarrow \Sigma_\sigma/\mathbf{Z}_p$: branched covering with b branch points

$y_1,\ \cdots,\ y_b$ of order $(n_1, \cdots, n_b)$

Then, $[G_p, G_p] \subset \ker\det(D_\ell,\ \cdot)$ $\Longrightarrow$

$$1 = \det(D_\ell, \gamma^\beta) = \det(D_\ell, g)^{m\beta} \iff \det(D_\ell, g)^d = 1$$

$$\iff 0 = dI_{D_\ell}(g^z) \equiv d\psi_{\ell,z}(t_1,\ \cdots,\ t_b) \quad (\mathrm{mod}\ \mathbf{Z})$$

$$\left(\begin{array}{l} \exists t_i : \text{ prime to } n_i \ (1 \le i \le b)\,, \\ \forall z \ (1 \le z < p,\ z : \text{prime to } p)\,,\ \forall \ell \ (0 \le \ell < p) \end{array} \right)$$

R-H equation, results of Harvey $\Longrightarrow$

$$(\sigma, p) = (2, 5) \Longrightarrow (b, \{n_1,\ \cdots,\ n_b\}) = (3, \{5, 5, 5\})$$

$$(\sigma, p) = (7, 5) \Longrightarrow (b, \{n_1,\ \cdots,\ n_b\}) = (3, \{5, 5, 5\})$$

$$(\sigma, p) = (3, 9) \Longrightarrow (b, \{n_1,\ \cdots,\ n_b\}) = (3, \{3, 9, 9\})$$

$$(\sigma, p) = (4, 9) \Longrightarrow (b, \{n_1,\ \cdots,\ n_b\}) = (3, \{9, 9, 9\})$$

$$(\sigma, p) = (11, 9) \Longrightarrow (b, \{n_1,\ \cdots,\ n_b\}) = (5, \{3, 9, 9, 9, 9\})$$

$$(\sigma, p) = (7, 10) \Longrightarrow (b, \{n_1,\ \cdots,\ n_b\}) = (4, \{2, 10, 10, 10\})\,,\ (5, \{2, 2, 2, 5, 10\})$$

$$(\sigma, p) = (5, 11) \Longrightarrow (b, \{n_1,\ \cdots,\ n_b\}) = (3, \{11, 11, 11\})\,.$$

Using Theorem 2, we have the following results.

$\underline{p = 5,\ \sigma = 2,\ 7 \implies (b, \{n_1, \cdots, n_b\}) = (3, \{5,5,5\})}$

$$\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z} \ \ (\forall t_1,\ t_2,\ t_3 = 1,\ 2,\ 3,\ 4) \implies G_5 \not\subset \mathrm{Aut}_J(\Sigma_2),\ \mathrm{Aut}_J(\Sigma_7)$$

$\underline{p = 9,\ \sigma = 3 \implies (b, \{n_1, \cdots, n_b\}) = (3, \{3,9,9\})}$

$$3\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z} \ \ (\forall t_1 = 1,\ 2\ ,\ \forall t_2,\ t_3 = 1,\ 2,\ 4,\ 5,\ 7,\ 8) \implies G_9 \not\subset \mathrm{Aut}_J(\Sigma_3)$$

$\underline{p = 9,\ \sigma = 4 \implies (b, \{n_1, \cdots, n_b\}) = (3, \{9,9,9\})}$

$$3\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z} \ \ (\forall t_1,\ t_2,\ t_3 = 1,\ 2,\ 4,\ 5,\ 7,\ 8) \implies G_9 \not\subset \mathrm{Aut}_J(\Sigma_4)$$

$\underline{p = 9,\ \sigma = 11 \implies (b, \{n_1, \cdots, n_b\}) = (5, \{3,9,9,9,9\})}$

$$3\psi_{1,1}(t_1, t_2, t_3, t_4, t_5) \notin \mathbf{Z} \ \ (\forall t_1 = 1,\ 2\ ,\ ,\ \forall t_2,\ t_3,\ t_4,\ t_5 = 1,\ 2,\ 3,\ 4)$$
$$\implies G_9 \not\subset \mathrm{Aut}_J(\Sigma_{11})$$

$\underline{p = 10,\ \sigma = 7 \implies (b, \{n_1, \cdots, n_b\}) = (4, \{2,10,10,10\}),\ (5, \{2,2,2,5,10\})}$

$$\begin{cases} k\psi_{1,1}(1, t_2, t_3, t_4) \notin \mathbf{Z}\ (k = 2,\ 5)\ \ (\forall t_2,\ t_3,\ t_4 = 1,\ 3,\ 7,\ 9) \\ k\psi_{1,1}(1,1,1, t_4, t_5) \notin \mathbf{Z}\ (k = 2,\ 5)\ \ (\forall t_4 = 1,\ 2,\ 3,\ 4,\ \forall t_5 = 1,\ 3,\ 7,\ 9) \end{cases}$$
$$\implies G_{10} \not\subset \mathrm{Aut}_J(\Sigma_7)$$

$\underline{p = 11,\ \sigma = 5 \implies (b, \{n_1, \cdots, n_b\}) = (3, \{11,11,11\})}$

$$\{(t_1, t_2, t_3) \mid \psi_{1,1}(t_1, t_2, t_3) \in \mathbf{Z}\} \cap \{(t_1, t_2, t_3) \mid \psi_{2,1}(t_1, t_2, t_3) \in \mathbf{Z}\} = \phi$$
$$\implies G_{11} \not\subset \mathrm{Aut}_J(\Sigma_5)$$

Hence $G_5 \not\subset \Gamma_2, \Gamma_7$, $G_9 \not\subset \Gamma_3, \Gamma_4, \Gamma_{11}$, $G_{10} \not\subset \Gamma_7$, $G_{11} \not\subset \Gamma_5$

In particular,

$G_p = D(2p)$: dihedral group generated by g, h with $g^{-1}h^{-1}gh = g^{p-2}$

g.c.d$(p, p-2) < p \implies$

$D(2p) \not\subset \Gamma_\sigma$ if $(\sigma, p) = (2,5),\ (7,5),\ (3,9),\ (4,9),\ (11,9),\ (7,10),\ (5,11)$

Note that $\sigma \equiv 0,\ 1 \pmod{p} \implies D(2p) \subset \Gamma_\sigma$.

3 0-pseudofree action of cyclic groups of prime order

Definition 1 An action is called "0-peudofree" iff the exceptional orbits are isolatec points. (E. Laitinen and P. Traczyk, Pseudofree representations and 2-pseudofree action: on spheres, Proc. Amer. Math. Soc. **97**, 151-157 (1986))
Here 0-pseudofree is simply called "pseudofree".

$\mathrm{Pf}(M)$: set of orientation-preserving isometries g on M
such that the fixed point set $\mathrm{Fix}(g)$ consists of points
When $M = M^{2m}$ has a Spin^c-structure,
$\mathrm{Pf}^c(M) := \{g \in \mathrm{Pf}(M) \mid g \text{ preserves the } \mathrm{Spin}^c\text{-structure}\}$
If $\mathbf{Z}_p = \langle g \rangle$, p : prime number
$\mathbf{Z}_p$-action is pseudo-free $\Longleftrightarrow$ $\mathbf{Z}_p \subset \mathrm{Pf}(M)$ $\Longleftrightarrow$ $\mathrm{Fix}(g) = \{q_1, q_2, \cdots, q_n\}$: points

Theorem 3 Assume that $\mathbf{Z}_p \subset \mathrm{Pf}(M)$ and suppose that $T_{q_i}M = \oplus_{j=1}^{m} N(\tau_{ij})$ $(1 \leq i \leq n)$. Then using $pI_D(g) = 0$ (D : signature operator), we have

$$\sum_{k=1}^{\frac{p-1}{2}} \sum_{i=1}^{n} \prod_{j=1}^{2s} \cot \frac{\pi k \tau_{ij}}{p} \equiv 0 \pmod{\mathbf{Z}} \quad \text{if } m = 2s$$

$$\sum_{k=1}^{\frac{p-1}{2}} \cot \frac{\pi k}{p} \sum_{i=1}^{n} \prod_{j=1}^{2s-1} \cot \frac{\pi k \tau_{ij}}{p} \equiv 0 \pmod{\mathbf{Z}} \quad \text{if } m = 2s-1$$

Corollary 1
$\mathbf{Z}_3 \subset \mathrm{Pf}(M) \Longrightarrow n$: even or $n \geq 3^{[(m+1)/2]}$

Assume that M has a Spin^c-structure and $\mathbf{Z}_p \subset \mathrm{Pf}^c(M)$
L : (arbitrary) $\mathbf{Z}_p$-line bundle, D_L : L-valued Dirac operator

Theorem 4 When $p = 2$, $n \geq 2^m$ unless n and $\mathrm{Ind}(D_L)$ $(\forall L)$ are even.

Theorem 5 When $p=3$ or $p=5$,

$$\gamma = \min_{s\in\mathbf{Z}} \left| \frac{p-1}{2}\mathrm{Ind}(D_L) - ps \right| \text{ : distance from } \frac{p-1}{2}\mathrm{Ind}(D_L) \text{ to } p\mathbf{Z}$$

$$\Longrightarrow n \geq \frac{\gamma}{3(p-1)}\left(2\sin\frac{\pi}{p}\right)^{m+1}$$

$$\det(D_L, g) = 1 \Longrightarrow n \geq \frac{\gamma}{p-1}\left(2\sin\frac{\pi}{p}\right)^{m+1}$$

Example 4

$$M_{pk} = \Sigma_{pk} \times \overbrace{S^2 \times \cdots \times S^2}^{m-1 \text{ times}} \quad (p : \text{prime})$$

has a Spin^c-structure with

$$c_1(\eta) = (2s+2-2pk)y + \sum_{j=1}^{m-1}(2t_j+2)z_j$$
$$\in H^2(M_{pk};\mathbf{Z}) \cong H^2(\Sigma_{pk};\mathbf{Z}) \oplus \oplus_{j=1}^{m-1} H^2(S^2;\mathbf{Z}) \quad (s,\ t_j \in \mathbf{Z})$$

M_{pk} admits an peudo-free Spin^c-action of $\mathbf{Z}_p$ with 2^m fixed points if the Spin^c-structure comes from the almost complex structure.

$\Longrightarrow$ s, t_j's equal 0.

Since we can show that

$$\mathrm{Ind}(D) = (s+1-pk)\prod_{j=1}^{m-1}(t_j+1) \equiv (s+1)\prod_{j=1}^{m-1}(t_j+1) \pmod p,$$

s, t_j's are even and $\mathbf{Z}_2 \subset \mathrm{Pf}^c(M_{2k}) \overset{\text{Theorem 4}}{\Longrightarrow} n \geq 2^m$

$$(s+1)\prod_{j=1}^{m-1}(t_j+1) \equiv 1,\ 2 \pmod 3,\ \mathbf{Z}_3 \subset \mathrm{Pf}^c(M_{3k})$$

$$\Longrightarrow \gamma = 1 \overset{\text{Theorem 5}}{\Longrightarrow} n \geq \frac{1}{6}3^{\frac{m+1}{2}} \ (\sim 0.866^m \cdot 2^m)$$

$$(s+1)\prod_{j=1}^{m-1}(t_j+1) \equiv 1,\ 4 \pmod 5,\ \mathbf{Z}_5 \subset \mathrm{Pf}^c(M_{5k})$$

$$\Longrightarrow \gamma = 2 \overset{\text{Theorem 5}}{\Longrightarrow} n \geq \frac{1}{6}\left(\frac{5-\sqrt{5}}{8}\right)^{\frac{m+1}{2}} \ (\sim 0.294^m \cdot 2^m)$$

$$(s+1)\prod_{j=1}^{m-1}(t_j+1) \equiv 2,\ 3 \pmod 5,\ \mathbf{Z}_5 \subset \mathrm{Pf}^c(M_{5k})$$

$$\Longrightarrow \gamma = 1 \overset{\text{Theorem 5}}{\Longrightarrow} n \geq \frac{1}{12}\left(\frac{5-\sqrt{5}}{8}\right)^{\frac{m+1}{2}} \ (\sim 0.294^m \cdot 2^m)$$

Moreover $\mathbf{Z}_3 \subset \mathrm{Pf}(M_{pk}) \overset{\text{Corollary 1}}{\Longrightarrow} n \neq 1, 3, 5, \cdots, 3^{[(m+1)/2]} - 2$

Example 5

$M = \mathbf{CP}^2 \times \mathbf{CP}^k \ (k \geq 3)$

Assume that $\mathrm{Pf}(M) \supset \mathbf{Z}_p = \langle g \rangle$ (p : odd prime)

$H^2(M;\mathbf{Z}) = H^2(\mathbf{CP}^2;\mathbf{Z}) \oplus H^2(\mathbf{CP}^k;\mathbf{Z}) = \{\lambda x + \mu y \mid \lambda, \mu \in \mathbf{Z}\} = \mathbf{Z} \oplus \mathbf{Z}$

where $H^2(\mathbf{CP}^2;\mathbf{Z}) \cong \mathbf{Z} = \langle x \rangle$ and $H^2(\mathbf{CP}^k;\mathbf{Z}) \cong \mathbf{Z} = \langle y \rangle$

$g^*(x^2y^k) = x^2y^k \overset{(*1)}{\Longleftrightarrow} g^*x = x\,,\ g^*y = y \overset{(*2)}{\Longrightarrow} \#\mathrm{Fix}(g) = 3(k+1)$

($(*1)$: not trivial, $(*2)$: Lefschetz fixed point theorem)

$$\left(\begin{array}{l} \text{For example, if } k < p, \\ \quad \text{pseudo-free action of } \mathbf{Z}_p \text{ on } \mathbf{CP}^j \ (j = 2 \text{ or } k) \text{ defined by} \\ \qquad g : [z_0 : z_1 : z_2 : \cdots : z_j] \longrightarrow [z_0 : \xi_p z_1 : \xi_p^2 z_2 : \cdots : \xi_p^j z_j] \\ \quad \text{has } j+1 \text{ fixed points} \\ \quad \Longrightarrow \ \text{diagonal action of } \mathbf{Z}_p \text{ on } M \text{ has } 3(k+1) \text{ fixed points} \end{array}\right)$$

Almost complex structure J of $M \Longrightarrow$ Spin^c-structure P_J of M

$\Longrightarrow c_1(\eta) = 3x + (k+1)y \Longrightarrow g^*c_1(\eta) = c_1(\eta)$

$H^1(M;\mathbf{Z}) = 0 \Longrightarrow \mathbf{Z}_p$-action lifts to an action on the Spin^c-structure P_J

We can show that $\mathrm{Ind}(D) = 1$, and hence

$$\gamma = 1 \ (p=3)\,,\ \gamma = 2 \ (p=5) \text{ in Theorem 5} \Longrightarrow$$
$$\begin{cases} 3(3-1)\cdot 3(k+1) \geq \left(2\sin\dfrac{\pi}{3}\right)^{2+k+1} \Longleftrightarrow k \leq 5 \quad (p=3) \\ 3(5-1)\cdot 3(k+1) \geq 2\left(2\sin\dfrac{\pi}{5}\right)^{2+k+1} \Longleftrightarrow k \leq 37 \quad (p=5) \end{cases}$$

$k \geq 3 \Longrightarrow 3(k+1) < 3^{[(3+k+1)/2]} \overset{\text{Corollary 1}}{\Longrightarrow} \mathbf{Z}_3 \not\subset \mathrm{Pf}(M)$ if k is even

If $G_p \subset \mathrm{Pf}^c(M)$ (p : odd prime), then $\det(D,g) = 1$ and hence

$$\gamma = 1 \ (p=3)\,,\ \gamma = 2 \ (p=5) \text{ in Theorem 5} \Longrightarrow$$
$$\begin{cases} (3-1)\cdot 3(k+1) \geq \left(2\sin\dfrac{\pi}{3}\right)^{2+k+1} \Longleftrightarrow \nexists\ k \quad (p=3) \\ (5-1)\cdot 3(k+1) \geq 2\left(2\sin\dfrac{\pi}{5}\right)^{2+k+1} \Longleftrightarrow k \leq 29 \quad (p=5) \end{cases}$$